B.Sc Physics Part-III

Paper - V, Group-A

(Mathematical Physics)

Cauchy-Riemann equations in polar form :—

In the last class note we have discussed Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{— (1)}$$

Eq. (1) provide necessary and sufficient conditions for a function $f(z) = u + iv$ in a region R. In the present class note we obtain the expression for Cauchy-Riemann equations in polar form —

Let (r, θ) be the coordinates of a point whose Cartesian coordinates are (x, y), therefore $z = x + iy = re^{i\theta}$ — (2)

and $u + iv = f(z) = f(re^{i\theta})$ — (3)

It is to be noted that u and v in eq. (3) are expressed in terms of r and θ. Now we obtain derivatives of u and v w.r.t r and θ.

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}).e^{i\theta}$$

and $$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})(ire^{i\theta}) = ir\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

or $$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r} \quad \text{— (4)}$$

Now equating real and imaginary parts, we obtain

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}} \quad \text{— (5)}$$

Polar form of Cauchy-Riemann equations.

H.W.: Using equation (5) obtain the expression

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$